Spherical collapse for dummies

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Abstract

With this I would like to collect and present in a simple and consistent form some of the various analytical derivation of the spherical collapse theory, in particular how to obtain the famous $\delta_{lin} \sim 1.686$.

Let's consider the equation of motion of a point on the surface of a sphere of matter: from the Gauss theorem it feels only the gravity due to the matter between it and the center of the sphere, so

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} = -\frac{GM(< R)}{R^2} \tag{1}$$

(R is the comoving coordinate).
Integrating once we obtain the conservation of energy:

$$\frac{\dot{R}^2}{2} - \frac{GM(< R)}{R} = const = E \tag{2}$$

with $M = \frac{4\pi R_i^3}{3}\bar{\rho}_i(1+\delta_i)$, i means "initial", $\delta_i = \frac{\int_0^{R_i} \mathrm{d}r 4\pi r^2 \delta_i(r)}{4\pi R_i^3/3}$ is the mean overdensity and $\Omega = \rho/\rho_c = \frac{8\pi G}{3}\frac{\rho}{H^2} = \frac{8\pi G}{3}\rho\frac{R_i^3}{H^2 R^3}$. Because $E < 0$ $\frac{\mathrm{d}R}{\mathrm{d}t}$ can change sign and the perturbation, initially expanding with the background, can collapse at later times.
We treat the sphere (that is our perturbation) and the background as two universes, with $\Omega > 1$ the former and $\Omega = 1$ the latter. We make some assumption for the spherical perturbation:

- it is homogeneous and isotropic
- peculiar velocities are zero in the initial conditions, that means we have only Hubble flux
- shell remain concentric, so we haven't shell crossing (in linear regime)

For both the sphere and the background we can write the Friedmann equation

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2 \tag{3}$$

and in the linear regime we have $\Delta_i \ll 1$ and peculiar velocities are zero, so

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)_i \sim \left(\frac{\mathrm{d}ax}{\mathrm{d}t}\right)_i = x_i \left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)_i = R_i \left[\frac{\mathrm{d}a}{\mathrm{d}t}\frac{1}{a}\right] = R_i \left(\frac{\dot{a}}{a}\right)_i = H_i R_i \tag{4}$$

where x is the physical coordinate.
The kinetic and potential energies can be now rewritten as

$$K_i = \frac{(H_i R_i)^2}{2} \tag{5}$$

$$W_i = -\frac{GM}{R_i} = -\frac{4\pi G R_i^3}{3r_i}\rho_i (1+\Delta_i) = -\frac{4\pi G\rho}{3}\frac{2}{2}R^2 (1+\Delta) \tag{6}$$

$$= -\Omega_i (1+\Delta_i)\frac{(H_i R_i)^2}{2} \tag{7}$$

remembering that $\Omega = \frac{8\pi G}{3}\frac{\rho}{H^2}$.
The total energy then become

$$E = K_i - W_i = K_i - K_i\Omega_i (1+\Delta_i) \tag{8}$$

and the collapse occur when $(1+\Delta_i) > \frac{1}{\Omega_i}$.
If the perturbation (read "the density contrast") is large enough respect to the background it expands and then, after reaching a maximum, it separate from the background and start to collapse. When it reaches the maximum (we call this moment "turn around" or TA) the kinetic energy is zero but the the total energy is conserved, so

$$E = -\frac{GM}{R_{TA}} = -\frac{R_i}{R_{TA}}K\Omega_i (1+\Delta_i) = E_i = K_i [1 - \Omega_i (1+\Delta_i)] \tag{9}$$

From the conservation of energy we obtain

$$\frac{R_{TA}}{R_i} = \frac{\Omega_i (1+\Delta_i)}{\Omega_i (1+\Delta_i) - 1} \tag{10}$$

If $\Omega_i = 1$ then $\frac{R_{TA}}{R_i} \sim \frac{1+\Delta_i}{\Delta_i} \sim \frac{1}{\Delta_i}$ ($\Delta_i \ll 1$ at TA) so R_{TA}/R_i depends only on Δ_i and not on the mass.
After the TA the perturbation experiences collapse, shell crossing and virialization.

Now we want to calculate the virial radius and density. To do this we consider the virial theorem and the energy conservation

$$-W_{vir} = 2K_{vir} \tag{11}$$
$$E = K_{vir} + W_{vir} \tag{12}$$

so

$$E = K_{vir} + W_{vir} = -\frac{W_{vir}}{2} = \frac{1}{2}\left[\Omega_i\left(1+\Delta_i\right)\frac{\left(H_i R_i\right)^2}{2}\right] \tag{13}$$
$$= \Omega_i\left(1+\Delta_i\right)\frac{H_i^2 R_i^2}{4} \sim \frac{1}{2}\frac{GM}{R_{vir}} = -\frac{GM}{R_{TA}} \tag{14}$$

so

$$R_{vir} \sim \frac{R_{TA}}{2} \tag{15}$$

and at the virialization the system is $2^3 = 8$ times denser then at TA.
Let's now describe the exact evolution, starting from the Friedmann equation

$$\dot{R}^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2 \tag{16}$$

and defining $\theta|\,\mathrm{d}t = \mathrm{d}\theta\frac{R(t)}{c}$. Substituting it in the previous expression we obtain

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 - \frac{8\pi G}{3}\rho R^2 = -kc^2 \tag{17}$$
$$\left[\frac{\mathrm{d}R}{\mathrm{d}\theta}\right]^2\frac{1}{R^2\left(t\right)} = -\frac{8\pi G}{3c^2} = -kc^2 \tag{18}$$
$$\left[\frac{\mathrm{d}R}{\mathrm{d}\theta}\right]^2 = R^2\left(t\right)\left[\frac{8\pi G}{3c^2} - kc^2\right] \tag{19}$$

If we assume no shell crossing we obtain mass conservation and substituting $\rho\left(t\right) = \rho_i\left(R_i/R\right)^3$ we have

$$\left[\frac{\mathrm{d}R}{\mathrm{d}\theta}\right]^2 \tag{20}$$

Introducing $R_\star = \frac{4\pi G}{3c^2}\rho_0 R_0^3 = \frac{GM}{c^2}$ the last equation becomes

$$\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{R}{R_\star}\right)\right]^2 = 2\frac{R}{R_\star} - k\left(\frac{R}{R_star}\right)^2 \tag{21}$$

If $k = 1$ and $\frac{R}{R_\star} = y$ we can rewrite the equation as $\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right) = 2y - y^2$ with known solution $y = \frac{R(t)}{R_\star} = 1 - \cos\theta$. From this we can obtain

$$t\left(\theta\right) = \int_0^\theta \mathrm{d}\theta' \frac{R\left(\theta'\right)}{c} = \frac{R_\star}{c} \int_0^\theta \mathrm{d}\theta' \left(1 - \cos\theta'\right) = \frac{R_\star}{c}\left(\theta - \sin\theta\right) \tag{22}$$

Combining the solutions for R and T we find the parametric equation of a cycloid.

Just a little digression. We have:

$$r = A\left(1 - \cos\theta\right) \sim A\frac{\theta^2}{2} \tag{23}$$

$$t = B\frac{\theta^3}{6} \tag{24}$$

$$\mathrm{d}t = B\left[\frac{\theta^2}{2} - \frac{\theta^4}{24}\right]\mathrm{d}\theta \tag{25}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{2}{B\theta^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} \tag{26}$$

and considering the equations of energy conservation and motion we obtain

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right) = \frac{GM}{r} + E \tag{27}$$

$$\frac{1}{2}\left[\frac{2}{B\theta^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right]^2 = \frac{GM^2}{A\theta^2} + E \tag{28}$$

$$\frac{2}{B^2\theta^2}\theta^2 A^2 = \frac{2GM}{A\theta^2} + \frac{E}{2}\frac{A^2}{B^2\theta^2} = \frac{GM}{A\theta^2} + \frac{E}{2} \tag{29}$$

$$\frac{A^3}{B^2} = \frac{GM}{A} + \frac{\theta^2}{2}E \tag{30}$$

that is probably wrong (according to Ravi Sheth solution) but we don't need it, and

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM}{r^2} \tag{31}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}r}{\mathrm{d}t}\right] = -\frac{GM}{r^2} \tag{32}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{2}{B\theta^2}\right]\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{GM}{r^2} \tag{33}$$

$$\frac{2}{B\theta^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{2}{B\theta^2}\theta A\right] = -\frac{GM}{A^2\frac{\theta^4}{4}} \tag{34}$$

$$\frac{2}{B\theta^2}2\frac{A}{B}\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{1}{\theta}\right] = -\frac{4GM}{A^2\theta^4} \tag{35}$$

$$-\frac{4A}{B^2\theta^2}\frac{1}{\theta^2} = -\frac{4GM}{A^2\theta^4} \tag{36}$$

$$\frac{A^3}{B^2} = GM \tag{37}$$

that seems to be correct.
At TA $\theta = \pi$ so $\frac{R_{TA}}{2R_i} = A$ that we substitute in B:

$$B = \frac{(AR_i)^{3/2}}{(GM)^{1/2}\, t_i} = \underbrace{\left(\frac{R_{TA}}{2R_i}\right)^{3/2}}_{A^{3/2}} R_i^{3/2}\frac{1}{(GM)^{1/2}\, t_i} \tag{38}$$

$$= \left(\frac{R_{TA}R_i}{2R_i}\right)^{3/2}\frac{1}{t}\left[\Omega_i\left(1+\Delta_i\right)\frac{(H_iR_i)^2}{2}\right]^{-1/2} \tag{39}$$

$$= \left(\frac{R_{TA}R_i}{2R_i}\right)^{3/2}\frac{1}{t}\left[\frac{2}{\Omega_i\left(1+\Delta_i\right)(H_iR_i)^2}\right]^{1/2} \tag{40}$$

$$= \left[\frac{1+1/\Delta}{2}\right]^{3/2}\frac{R_i^{3/2}}{t_i}\frac{\sqrt{2}}{H_iR_i}\frac{1}{\left[\Omega\left(1+\Delta\right)\right]^{1/2}} \tag{41}$$

$$= \frac{1+1/\Delta}{2}\frac{R_i^{3/2}}{t_i}\frac{1}{H_iR_i\Omega^{1/2}} \tag{42}$$

From the energy conservation

$$\frac{R_{TA}}{R_i} = \frac{\Omega_i\left(1+\Delta_i\right)}{\Omega_i\left(1+\Delta_i\right)-1} \tag{43}$$

so

$$R_i = \Omega_i\left(1+\Delta_i\right) \tag{44}$$

and we find

$$B=\frac{1+1/\Delta_i}{t_iR_iH_i}\left[\Omega_i\left(1+\Delta_i\right)-1\right]^{3/2}=\frac{1+1/\Delta_i}{t_iR_iH_i}\left[1+\Delta_i-\frac{1}{\Omega_i}\right]^{-3/2} \tag{45}$$

Let's consider $\Omega=1$ (EdS model) and $\bar{\rho}=\frac{1}{6\pi Gt^2}$, we find

$$1+\Delta=\frac{\bar{\rho}_i}{\bar{\rho}\left(t\right)}\left(\frac{R_i}{R}\right)^3\sim\frac{\left(t/t_i\right)^2}{A^3\left(1-\cos\theta\right)^3}=\frac{B^2\left(\theta-\sin\theta\right)^2}{A^3\left(1-\cos\theta\right)^3} \tag{46}$$

$$=\frac{\rho\left(t\right)}{\bar{\rho}\left(t\right)}=\frac{\frac{3M_0}{4\pi R^3(\theta)}}{\frac{1}{6\pi Gt^2(\theta)}}=\underbrace{\frac{GM_0}{c^2R_\star}}_{=1\text{ because of }R_\star}\frac{9}{2}\frac{\left(\theta-\sin\theta\right)^2}{A^3\left(1-\cos\theta\right)^3} \tag{47}$$

Now, Taylor expanding sin and cos

$$\left(\theta-\sin\theta\right)^2\sim\left[\theta-\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}\right)\right]^2 \tag{48}$$

$$\left(1-\cos\theta\right)^3\sim\left[1-\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{4!}\right)\right]^3 \tag{49}$$

and substitute, obtaining

$$1+\Delta=\frac{9}{2}\frac{\left(\theta-\sin\theta\right)^2}{\left(1-\cos\theta\right)^3}=\frac{9}{2}\frac{\left(\frac{\theta^3}{6}-\frac{\theta^5}{120}\right)^2}{\left(\frac{\theta^2}{2}-\frac{\theta^4}{4}\right)^3}=\frac{9}{2}\frac{\frac{\theta^6}{36}\left(1-\frac{\theta^2}{5}\right)^2}{\frac{\theta^6}{8}\left(1-\frac{\theta^2}{4}\right)^3} \tag{50}$$

$$\sim\frac{9}{2}\frac{2}{9}\left(1-\frac{\theta^2}{10}\right)\left(1+\frac{\theta^2}{4}\right)\sim 1-\frac{3}{20}\theta^2 \tag{51}$$

so that

$$\Delta\left(\theta\right)\sim\frac{3}{20}\theta^2 \tag{52}$$

Before we found $t\left(\theta\right)=\frac{R_\star}{c}\left(\theta-\sin\theta\right)\sim\frac{R_\star}{c}\frac{\theta^3}{6}$, substituting we obtain

$$\Delta\left(t\right)=\frac{3}{20}\left[\frac{6ct}{R_\star}\right]^{2/3} \tag{53}$$

At TA $\theta=\pi$ and $t=\pi R_\star/c$ so $\Delta=\frac{3}{20}\left(6\pi\right)^{2/3}=1.06$ while the real density contrast is $\Delta_{TA}=\frac{9}{2}\frac{\pi^2}{8}-1=4.55$.
At the collapse we have

$$\theta=2\pi \tag{54}$$

$$t=\frac{2\pi R_\star}{c} \tag{55}$$

$$\Delta=\Delta_{lin}=\frac{3}{20}\left(12\pi\right)^{2/3}=1.686\ldots \tag{56}$$

while the real density contrast is

$$1 + \Delta_{vir} = \frac{9\pi^2}{16} \left[\frac{R_{TA}}{R_{vir}} \right]^3 \left[\frac{\bar{\rho}_{TA}}{\bar{\rho}_{vir}} \right] = \frac{9\pi^2}{16} \times 8 \times 4 \sim 178 \tag{57}$$

References

- Ravi Sheth, “Sperical evolution model”
- Bepi Tormen, “Formazione delle strutture cosmiche” - chapter by Porciani
- Mo, van den Bosh and White, “Galaxy formation and evolution”

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